Submodular Functions:
Finding influencers in networks and Detecting disease outbreaks

CS345a: Data Mining
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Motivation (1)

- Feature selection:
  - Given a set of features $X_1, \ldots X_n$
  - Want to predict Y from a subset $A = (X_{i1},\ldots,X_{ik})$
  - What are the k most informative features?
- Active learning:
  - Want to predict medical condition
  - Each test has a cost (but also reveals information)
  - Which tests should we perform to make most effective decisions?

Motivation (2)

- Influence maximization:
  - In a social network, which nodes to advertise to?
  - Which are the most influential blogs?
- Sensor placement:
  - Given a water distribution network
  - Where should we place sensors to quickly detect contaminations?

Combinatorial formulation

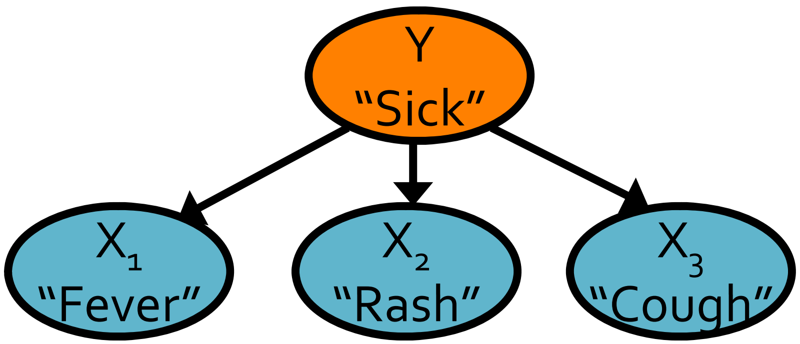
- Given:
  - finite set V
  - A function F: $2^V \rightarrow \Re$
- Want:

  $A^* = \text{argmax}_A \, F(A)$

  s.t. some constraints on A
- For example:
  - Influence maximization: V= F(A)=
  - Sensor placement: V= F(A)=
  - Feature selection: V= F(A)=

Example: Feature selection

- Given random variables Y, X_1, … X_n
- Want to predict Y from subset A = $(X_{i1},\ldots,X_{ik})$



Naïve Bayes Model:
$P(Y,X_1,\ldots,X_n)$
$= P(Y)\ \prod_i P(X_i\,|\,Y)$

- Want k most informative features:

$$A^* = \text{argmax } I(A;\, Y) \text{ s.t. } |A| \leq k$$

where $I(A;\, Y) = H(Y) - H(Y \mid A)$

Uncertainty before knowing A

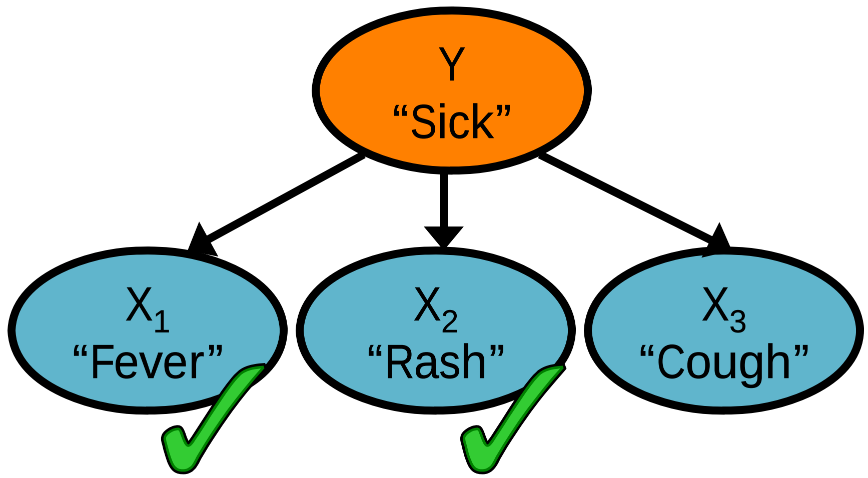
Uncertainty after knowing A

Example: Feature selection

- **Given:** finite set V of features, utility function $F(A) = I(A; Y)$
- **Want:** $A^* \subseteq V$ such that

$$\mathcal{A}^* = \operatorname*{argmax}_{|\mathcal{A}| \leq k} F(\mathcal{A})$$

Typically NP-hard!



Greedy hill-climbing:

Start with $A_0 = \{\}$
For i = 1 to k
$s^* = \text{argmax}_s F(A \cup \{s\})$
$A_i = A_{i-1} \cup \{s^*\}$

How well does this simple heuristic do?

Approximation guarantee

- Greedy hill climbing produces a solution A where F(A) $\geq$(1-1/e) of optimal value (~63%)
  [Hemhauser, Fisher, Wolsey '78]

- Claim holds for functions *F* with 2 properties:
  - *F* is monotone:
    if A $\subseteq$ B then F(A) $\leq$ F(B) and *F*({})=0
  - *F* is submodular:
    adding element to a set gives less improvement than adding to one of subsets

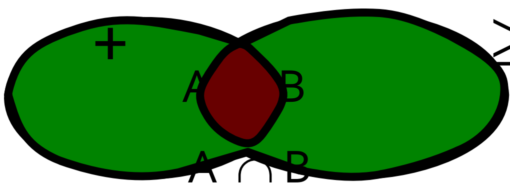
Submodularity: Definition 1

Definition:

- Set function F on V is called submodular if:

  For all $A,B \subseteq V$:

$$F(A)+F(B) \geq F(A \cup B)+F(A \cap B)$$



$+$ $\geq$ $+$

Submodularity: Or equivalently

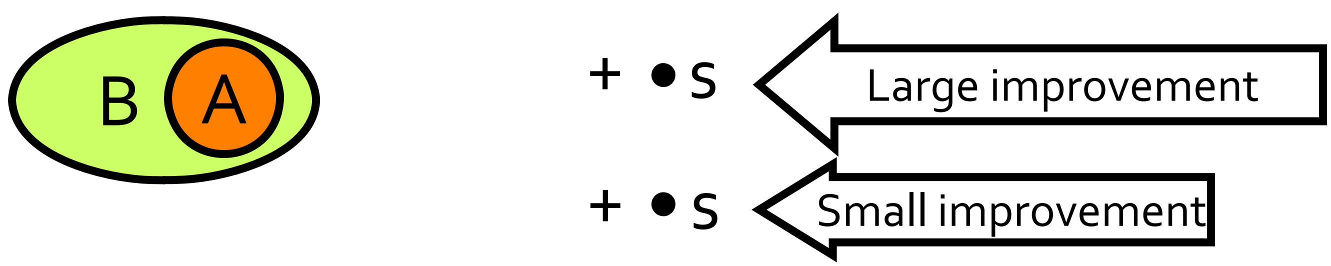
- Diminishing returns characterization

Definition:

- Set function F on V is called submodular if:

  For all $A \subseteq B$, $s \notin B$:

$$\underbrace{F(A \cup \{s\}) - F(A)}_{\text{Gain of adding } s \text{ to a small set}} \geq \underbrace{F(B \cup \{s\}) - F(B)}_{\text{Gain of adding } s \text{ to a large set}}$$



Example: Feature selection

- Given random variables $X_1,\ldots,X_n$
- Mutual information:

  $F(A) = I(A; V\backslash A) = H(V\backslash A) - H(V\backslash A|A)$

  $= \sum_{y,A} P(A)\,[\log P(y|A) - \log P(y)]$

  $H(C|D) = H(C,D) - H(D)$
- Mutual information F(A) is submodular

  [Krause-Guestrin '05]

  $F(A \cup \{s\}) - F(A) = H(s|A) - H(s|\, V\backslash(A\cup\{s\}))$
  - $A \subseteq B \Rightarrow H(s|A) \geq H(s|B)$ $\quad B = A \cup \{\ldots\}$
    - "Information never hurts"

Example: Feature selection (2)

- Let $Y = \sum_i \alpha_i X_i + \varepsilon$, where $(X_1,\ldots,X_n,\varepsilon) \sim N(\cdot;\, \mu,\Sigma)$
- Want to pick a subset A to predict Y
- $Var(Y \mid X_A=x_A)$: conditional var. of Y given $X_A=x_A$
- Expected variance:

  $$Var(Y \mid X_A) = \int p(x_A)\, Var(Y \mid X_A=x_A)\, dx_A$$

- Variance reduction:

  $$F_V(A) = Var(Y) - Var(Y \mid X_A)$$

- Then [Das-Kempe 08]:
  - $F_V(A)$ is monotonic
  - $F_V(A)$ is submodular*

    *under some conditions on Σ

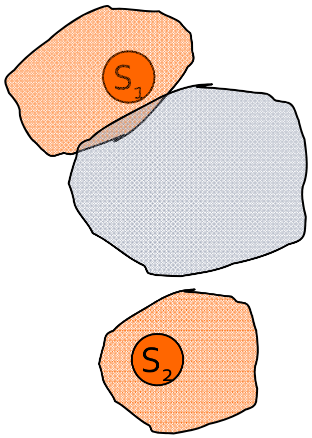
Orthogonal matching pursuit [Tropp-Donoho] near optimal!

Closedness properties

- $F_1,\ldots,F_m$ submodular functions on V and $\lambda_1,\ldots,\lambda_m > 0$
- Then: $F(A) = \sum_i \lambda_i F_i(A)$ is submodular!
- Submodularity closed under nonnegative linear combinations
- Extremely useful fact:
  - $F_\theta(A)$ submodular $\Rightarrow \sum_\theta P(\theta)\, F_\theta(A)$ submodular!
  - Multicriterion optimization:
    $F_1,\ldots,F_m$ submodular, $\lambda_i>0 \Rightarrow \sum_i \lambda_i F_i(A)$ submodular

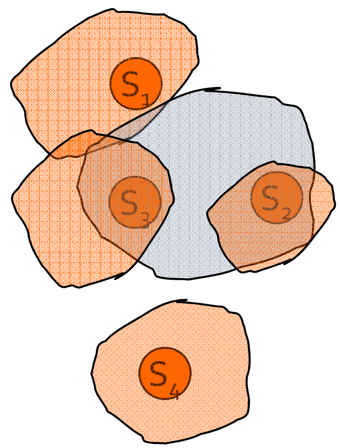
Example: Set cover

- Each element covers some area
- Observation: Diminishing returns



New element:

S'



$A=\{S_1, S_2\}$

$B=\{S_1, S_2, S_3, S_4\}$

Adding S' helps a lot

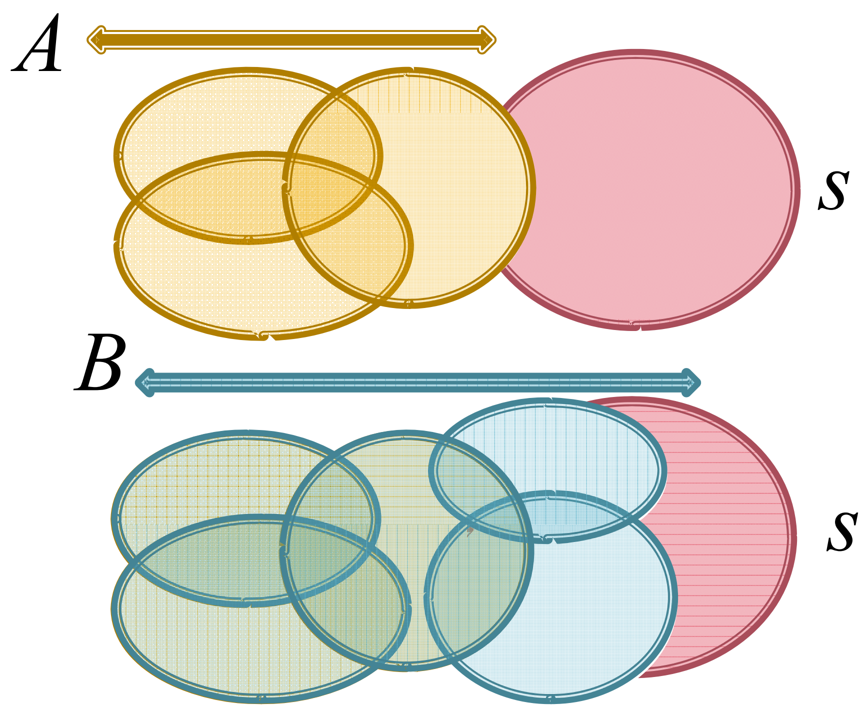
Adding S' helps very little

Example: Set cover

- F is submodular: $\boldsymbol{A \subseteq B}$

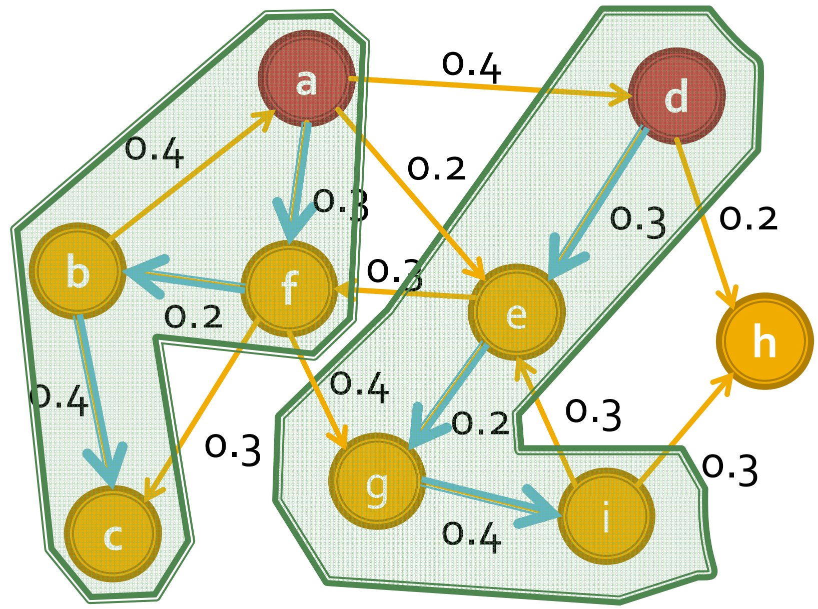
$$\underbrace{F(A \cup \{s\}) - F(A)}_{\text{Gain of adding a set s to a small solution}} \geq \underbrace{F(B \cup \{s\}) - F(B)}_{\text{Gain of adding a set s to a large solution}}$$

- Natural example:
  - Sets $s_1, s_2, \dots, s_n$
  - F(A) = size of union of s_i (size of covered area)



Example: Influence maximization

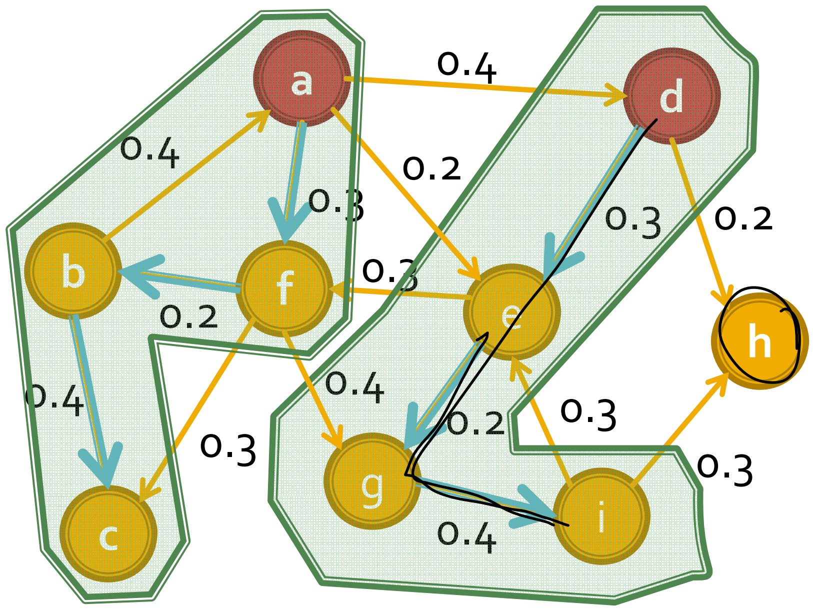
- Most influential set of size k: set S of k nodes producing largest expected cascade size F(S) if activated [Domingos-Richardson '01]



- Optimization problem: $$\max_{\text{S of size k}} F(S)$$

Influence maximization

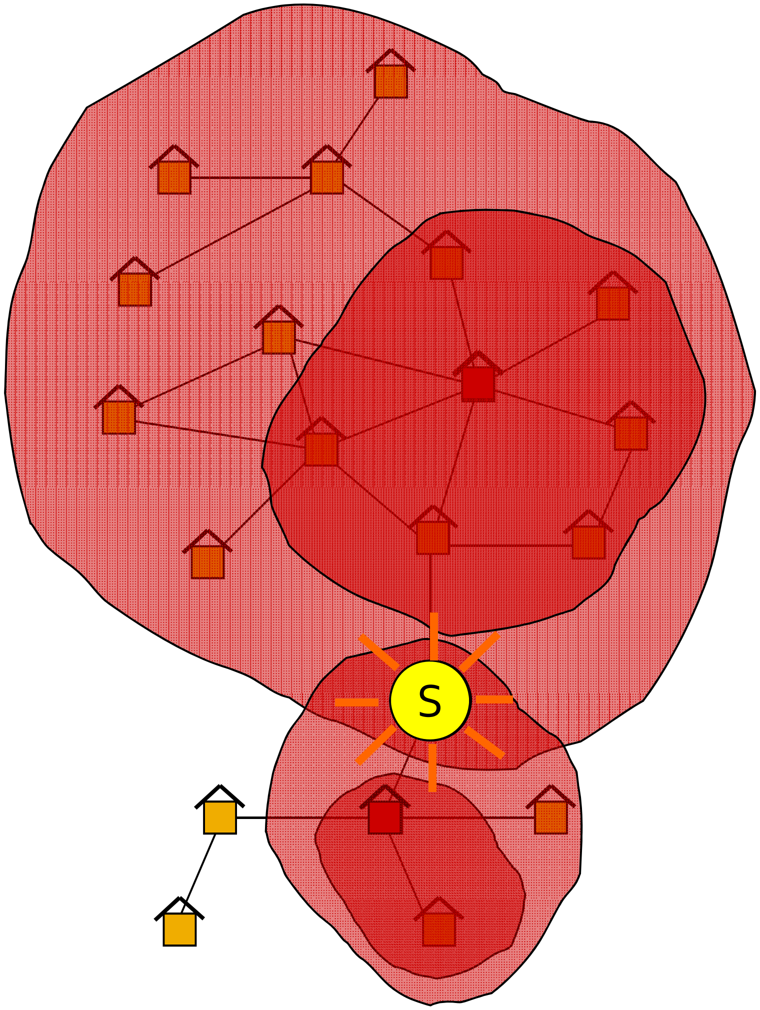
- Fix outcome *i* of coin flips
- Let $F_i(S)$ be size of cascade from *S* given these coin flips



- Let $F_i(v)$ = set of nodes reachable from *v* on live-edge paths
- $F_i(S)$ = size of union $F_i(v)$ → F_i is submodular
- $F = \sum F_i$ → *F* is submodular [Kempe-Kleinberg-Tardos '03]

Example 2: Water Network

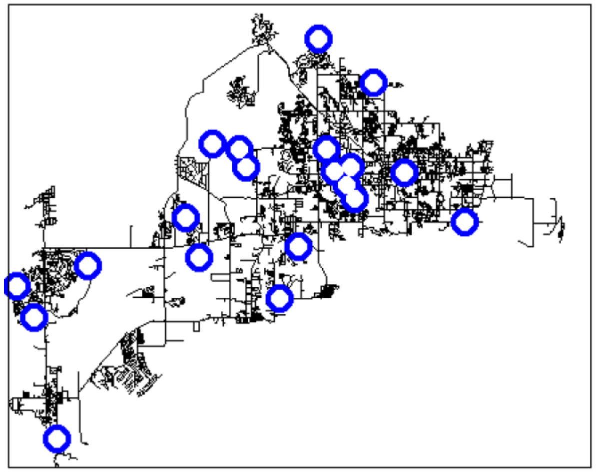
- Given a real city water distribution network
- And data on how contaminants spread in the network
- Problem posed by *US Environmental Protection Agency*



[Leskovec et al., KDD ’07]

Water Network

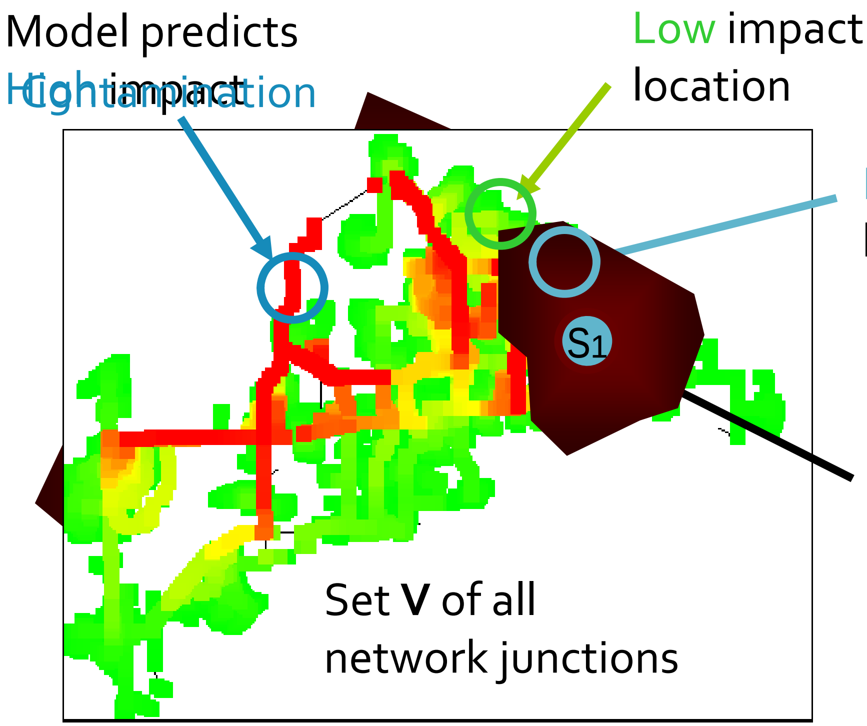
- Real metropolitan area water network:
  - V = 21,000 nodes
  - E = 25,000 pipes



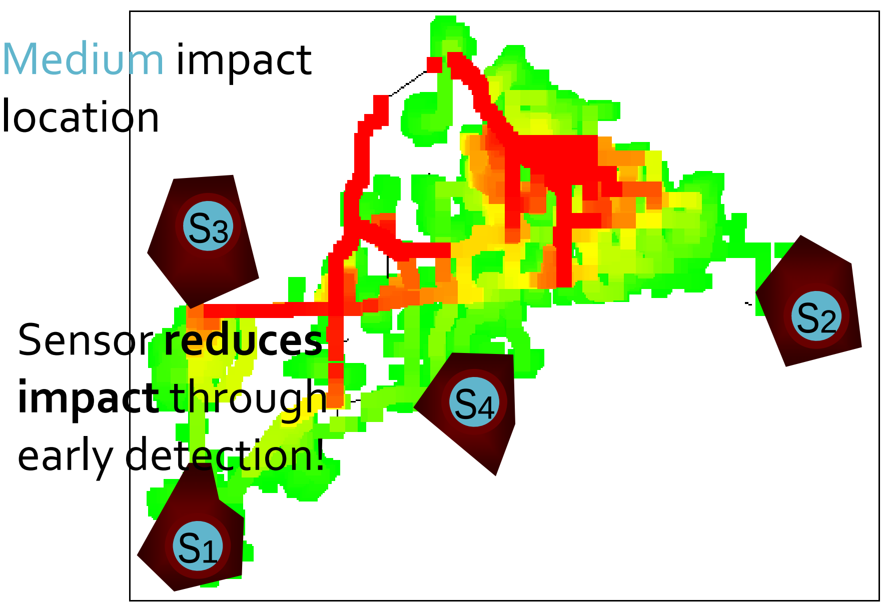
- Water flow simulator provided by EPA
- 3.6 million contamination events
- Multiple objectives:
  - Detection time, affected population, …
- Place sensors that detect well “on average”

Water Network: Utility

- Utility of placing sensors
  - Water flow dynamics, demands of households, …
- For each subset $A \subseteq V$ compute utility $F(A)$



High sensing quality F(A) = 0.9



Low sensing quality F(A)=0.01

Optimization problem

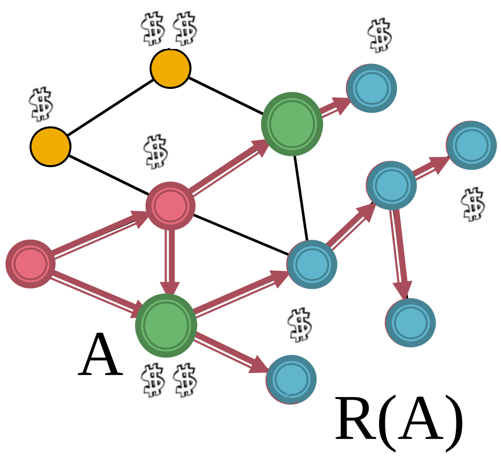
- Given:
  - Graph G(V,E), budget B
  - Data on how outbreaks $o_{1,} \ldots, o_i, \ldots, o_K$ spread over time
- Select a set of nodes A maximizing the reward

$$\max_{A \subseteq V} \sum_i \mathrm{Prob}(i) \underbrace{R_i(A)}_{\text{Reward for detecting outbreak } i}$$

subject to $cost(A) \leq B$

Two parts to the problem

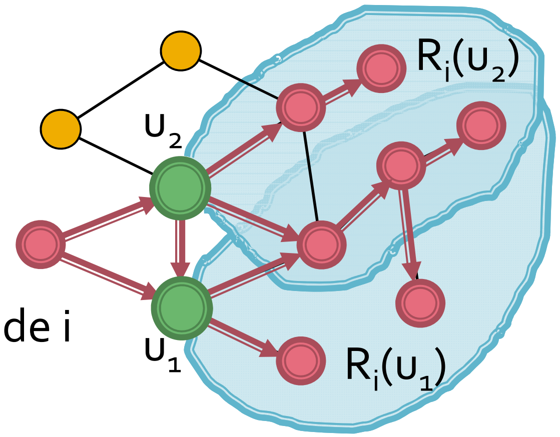
- Cost:
  - Cost of monitoring is node dependent
- Reward:
  - Minimize the number of affected nodes:
    - If A are the monitored nodes, let $R(A)$ denote the number of nodes we save



Reward function is submodular

- <u>Claim:</u> [Krause et al. '08]
  - Reward function is submodular
- Consider cascade *i*:
  - $R_i(u_k)$ = set of nodes saved from u_k
  - $R_i(A)$ = size of union $R_i(u_k)$, $u_k \in A$

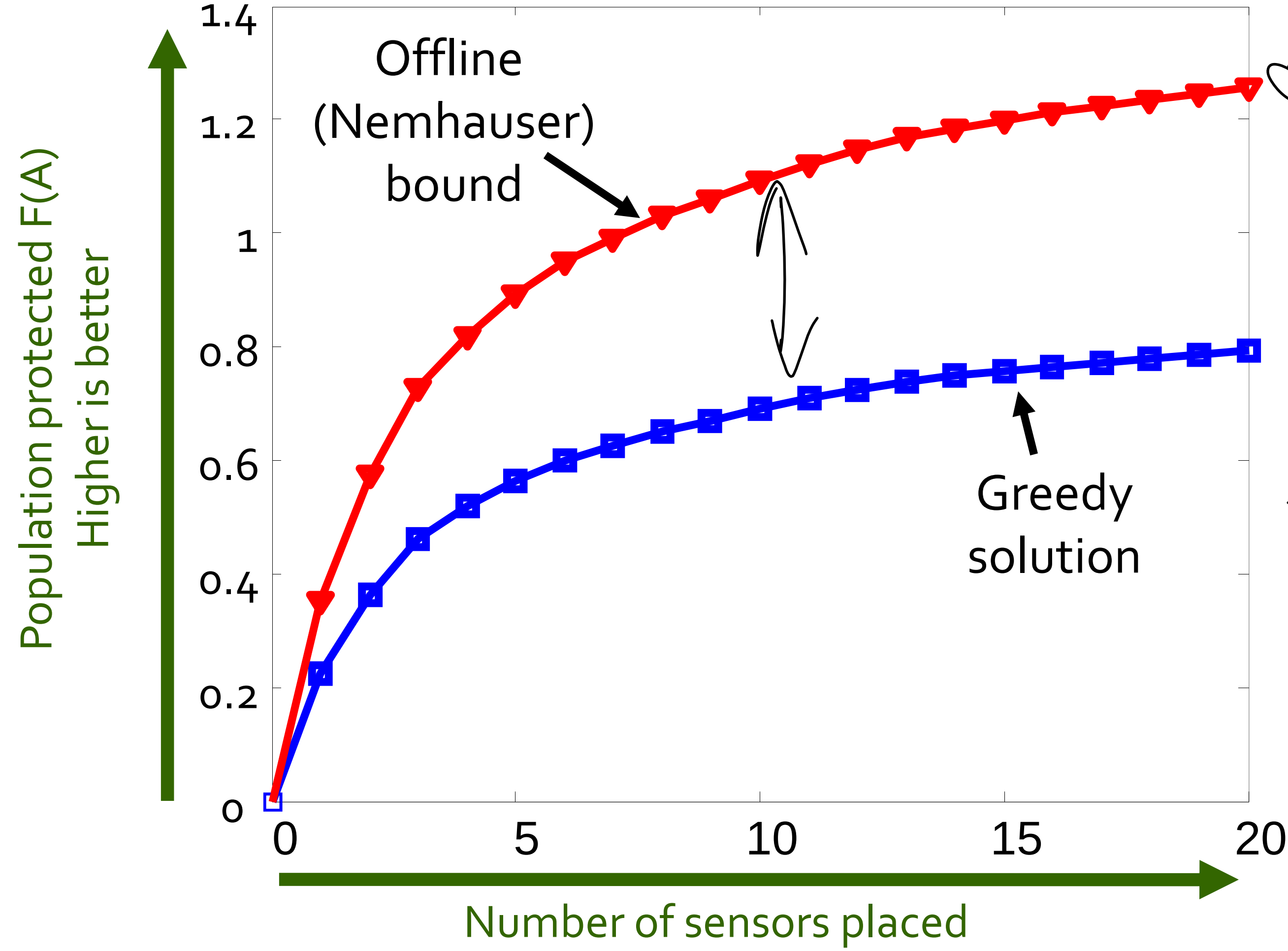
  $\Rightarrow R_i$ is submodular



- Global optimization:
  - $R(A) = \sum \text{Prob}(i)\, R_i(A)$

  $\Rightarrow$ R is submodular

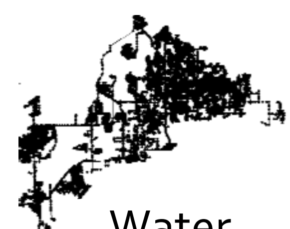
Solution quality: Nemhauser



OPT $\frac{1}{0.63}$

$(1-\frac{1}{e})\cdot OPT$

0.63



Water network

(1-1/e) bound quite loose… can we get better bounds?

Solution quality: Better estimate

- Suppose A is some solution to
  $$\arg\max_A F(A) \text{ s.t. } |A| \le k$$
  and $A^* = \{s_1,\dots,s_k\}$ is OPT solution
- Then:

So: For each $s \in V \setminus A$,
let $\delta_s = F(A \cup \{s\}) - F(A)$
Order so that $\delta_1 \ge \delta_2 \ge \dots \ge \delta_n$
Then: $F(A^*) \le F(A) + \sum_{i=1}^{k} \delta_i$

$$F(A^*) \le F(A \cup A^*)$$
$$= F(A) + \sum_{i=1}^{k} F(A \cup \{s_1 .. s_i\}) - F(A \cup \{s_1 \dots s_{i-1}\})$$
$$F(A \cup \{s_1 .. s_{i-1}\} \cup \{s_i\}) - F(A \cup \{s_1 \dots s_{i-1}\})$$

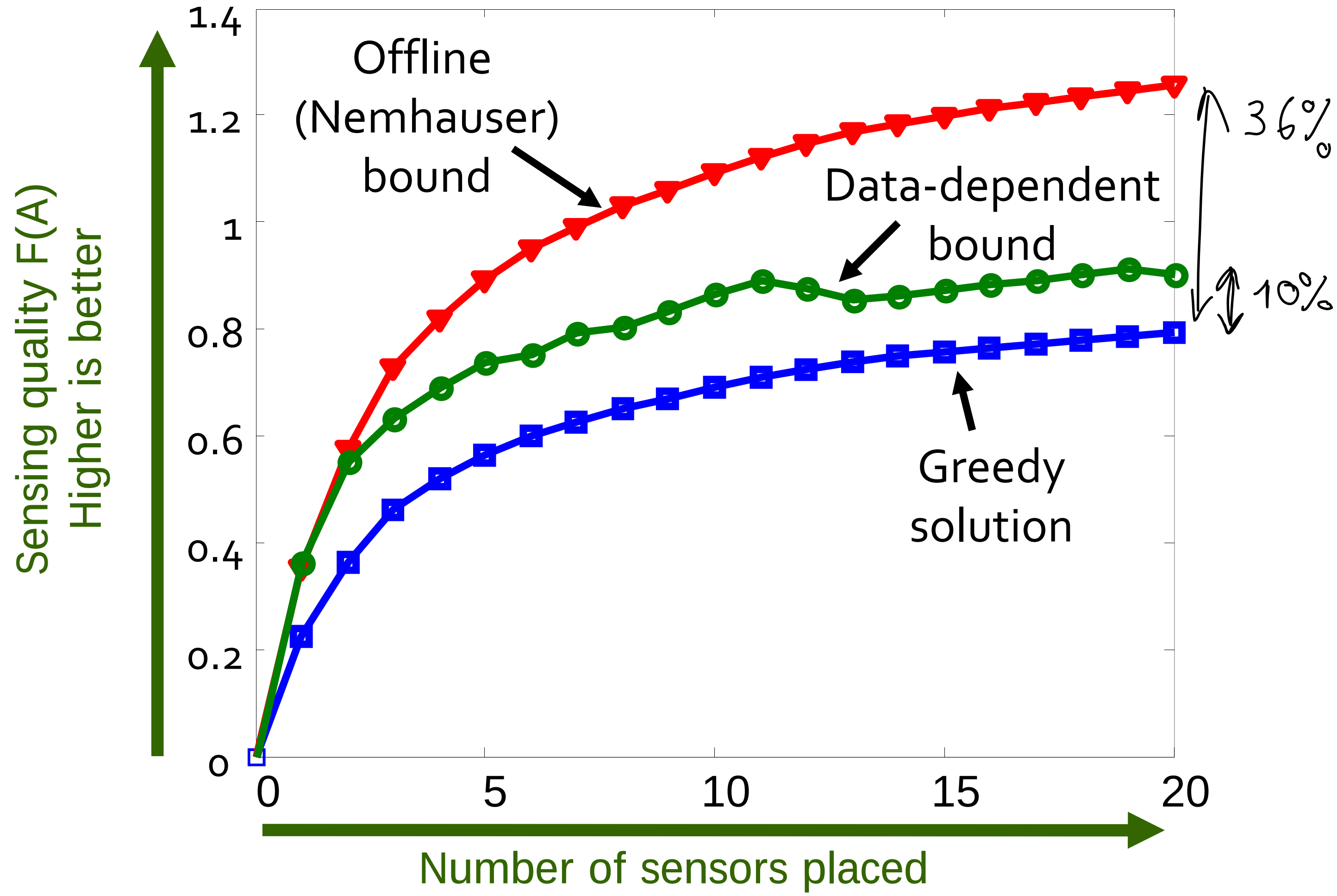
$$\delta_i = F(A \cup s_i) - F(A)$$

$$\delta_1 > \delta_2 > \dots > \delta_n$$

$$\le F(A) + \sum_{i=1}^{k} \underbrace{\left[F(A \cup \{s_i\}) - F(A)\right]}_{\delta_{s_i}}$$
$$= F(A) + \sum_{i=1}^{k} \delta_{s_i}$$

$$F(A^*) \le F(A) + \sum_{i=1}^{k} \delta_{s_i} \le F(A) + \sum_{i}^{k} \delta_i$$

Bounds on optimal solution



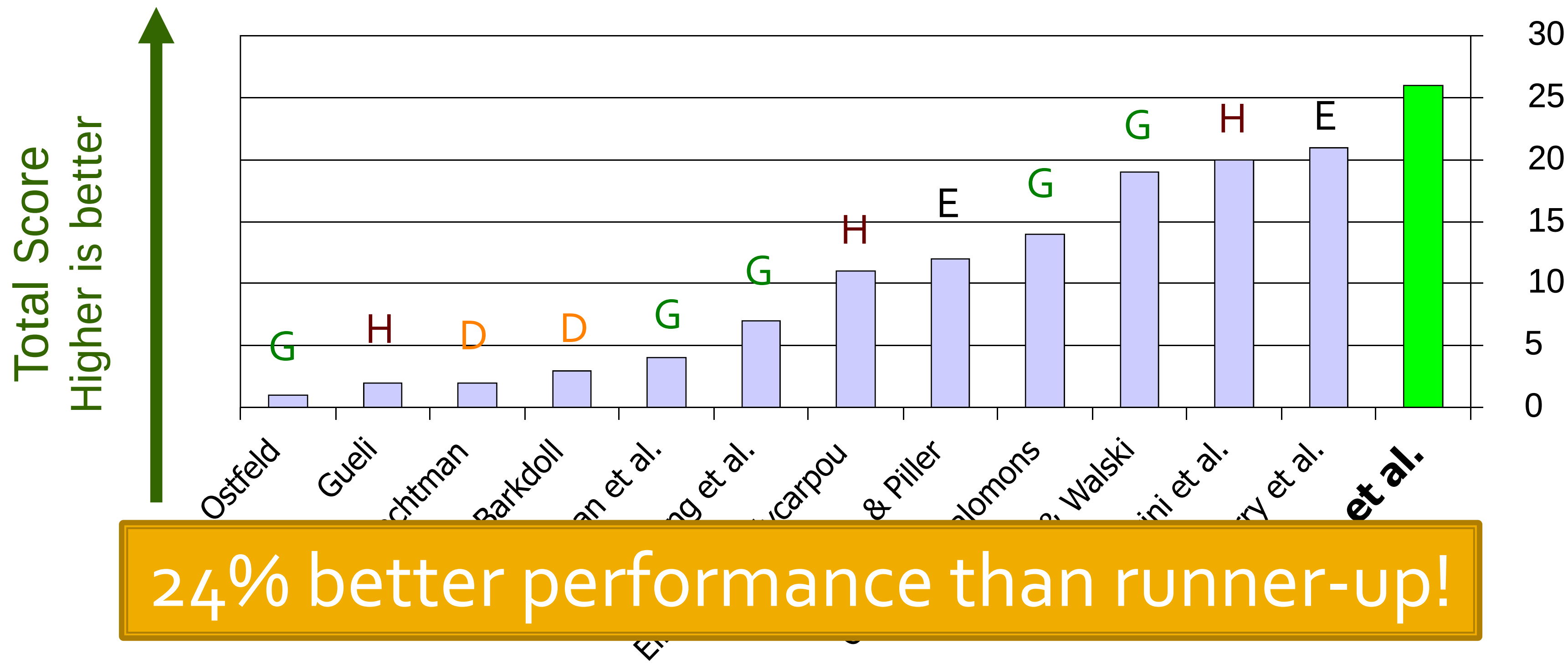
Submodularity gives **data-dependent** bounds on the performance of **any** algorithm

BWSN Competition results

- 13 participants
- Performance measured in 30 different criteria

G: Genetic algorithm D: Domain knowledge

H: Other heuristic E: "Exact" method (MIP)



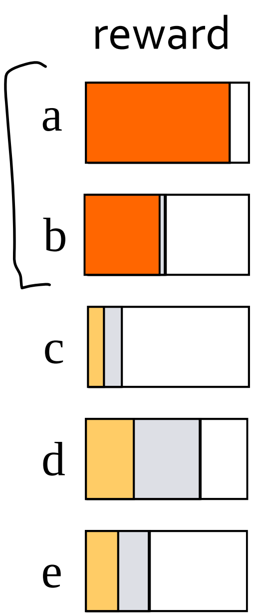
24% better performance than runner-up!

What was the trick?

- Simulated 3.6M contaminations on 40 machines for 2 weeks [Krause et al. ‘08]
  - 152 GB of simulation data
  - 16GB in RAM (compressed)
- Very accurate computation of F(A)
- Very slow evaluation of F(A):
  - Would take 6 weeks for all 30 settings

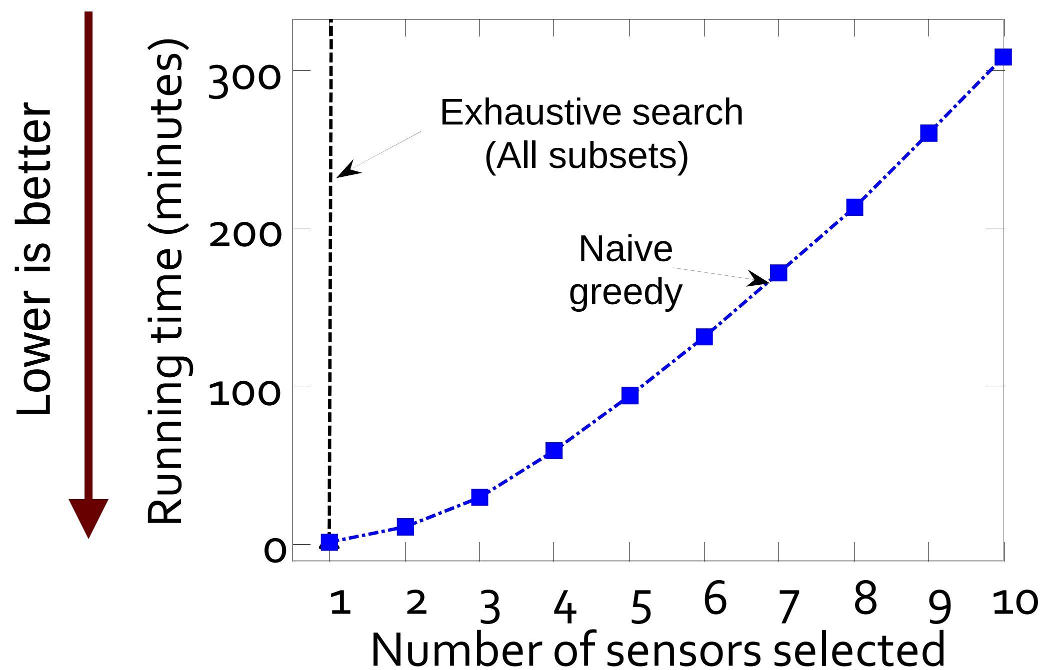
Greedy hill climbing

Hill-climbing



Add element with highest marginal gain

- Hill-climbing algorithm is slow:
  - At each iteration we need to re-evaluate gains of all sensors
  - It scales as $O(n \cdot k)$

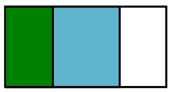


Scaling up greedy algorithm

- In round i+1:
  - have so far picked $A_i = \{s_1,\ldots,s_i\}$
  - pick $s_{i+1} = \text{argmax}_s\ F(A_i \cup \{s\})\text{-}F(A_i)$
    i.e., maximize "marginal benefit" $\delta_s(A_i)$
    $\delta_s(A_i) = F(A_i \cup \{s\})\text{-}F(A_i)$
    $\delta_s(A_i \cup C) = F(A \cup C \cup \{s\}) - F(A \cup C)$
- **Observation:** Submodularity implies
  $i \leq j \Rightarrow \delta_s(A_i) \geq \delta_s(A_j)$

$A_i \leq A_j$

$\delta_s(A_i) \geq \delta_s(A_{i+1})$

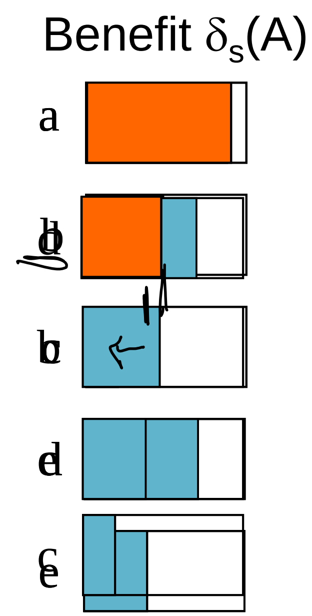
s 

Marginal benefits δ_s never increase!

"Lazy" greedy algorithm

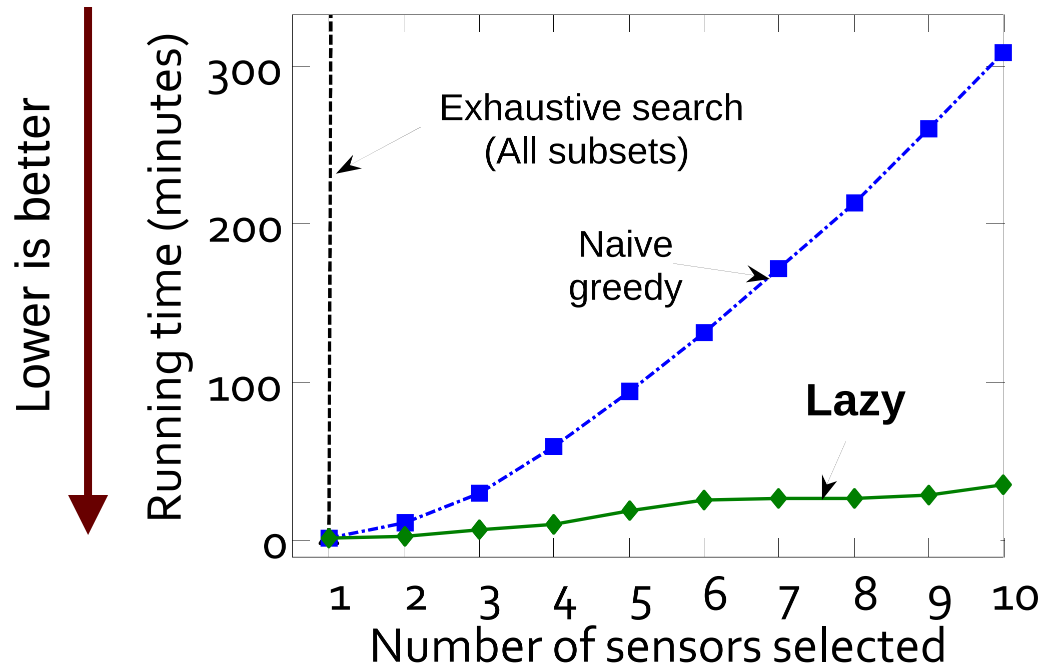
Lazy hill climbing algorithm:

- First iteration as usual
- Keep an ordered list of marginal benefits δ_i from previous iteration
- Re-evaluate δ_i only for top element
- If δ_i stays on top, use it, otherwise re-sort



Result of lazy evaluation

- Using "lazy evaluations" [Krause et al. '08]
  - 1 hour/20 sensors
- Done in 2 days!



Non-constant cost functions

- For each $s \in V$, let $c(s) > 0$ be its cost (e.g., feature acquisition costs, …)
- Cost of a set $C(A) = \sum_{s \in A} c(s)$ (modular function)
- Want to solve:

  $A^* = \text{argmax}_A F(A)$ s.t. $C(A) \leq B$ (budget)

- **Cost-benefit greedy algorithm:**

  Start with $A = \{\}$

  While there is an $s \in V \backslash A$ s.t. $C(A \cup \{s\}) \leq B$

  $$s^* = \underset{s:C(\mathcal{A}\cup\{s\})\leq B}{\text{argmax}} \frac{F(\mathcal{A}\cup\{s\}) - F(\mathcal{A})}{c(s)}$$

  $A = A \cup \{s^*\}$

Performance of cost-benefit greedy

- Consider the following problem:

  $$\max_A F(A) \text{ s.t. } C(A) \leq 1$$

Set A	F(A)	C(A)
{a}	2ε	ε
{b}	1	1

Cost-benefit greedy picks a.
Then cannot afford b!

➔ Cost-benefit greedy performs arbitrarily badly!

Cost-benefit optimization

- Theorem [Leskovec-Krause et al. '07]:
  - A_{CB}: cost-benefit greedy solution and
  - A_{UC}: unit-cost greedy solution (i.e., ignore costs)

  Then:

  $$\max \{ F(A_{CB}), F(A_{UC}) \} \geq ½ (1-1/e) \text{ OPT}$$

- Can still compute online bounds and speed up using lazy evaluations

- Note: Can also get
  - (1-1/e) approximation in time $O(n^4)$ [Sviridenko '04]
  - Slightly better than ½(1-1/e) in $O(n^2)$ [Wolsey '82]

Question...

TIME
IN PARTNERSHIP WITH CNN

Thursday, Nov. 20, 2008

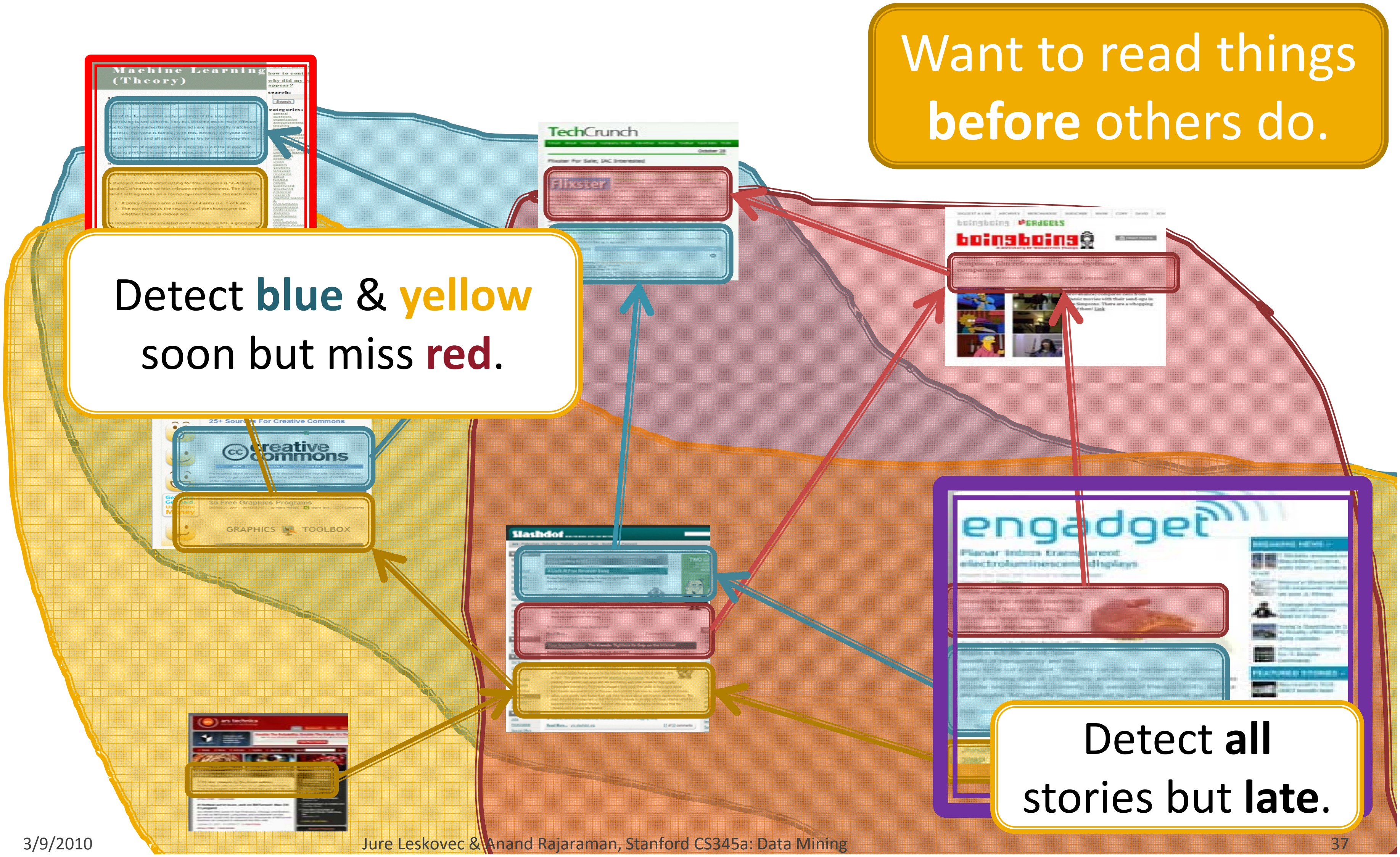
How Many Blogs Does the World Need?

By Michael Kinsley

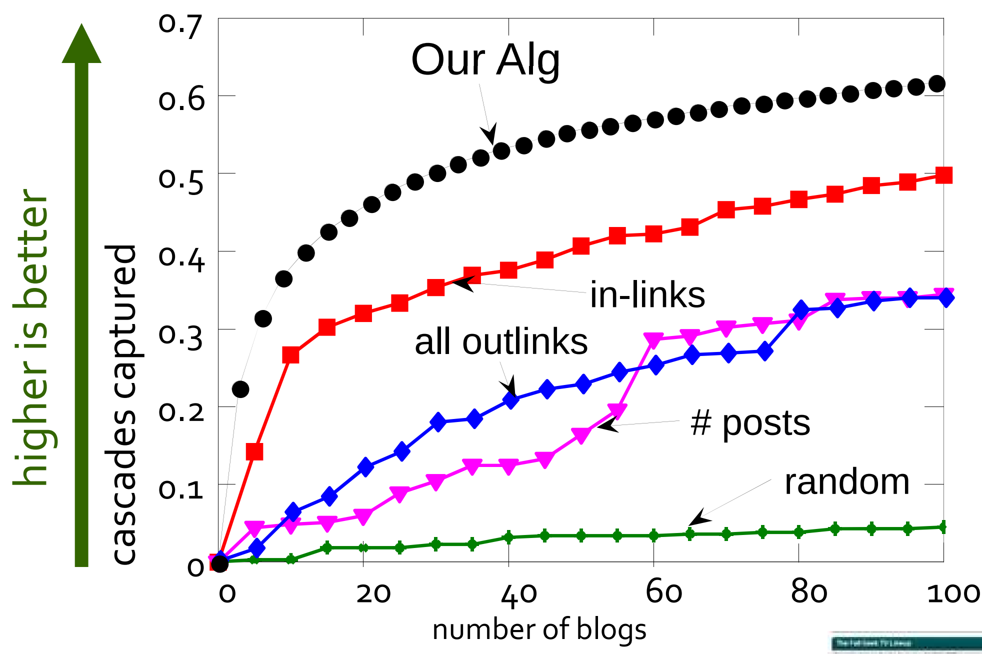
- I have 10 minutes. Which blogs should I read to be most up to date?
  [Leskovec-Krause et al. '07]

- Who are the most influential bloggers?

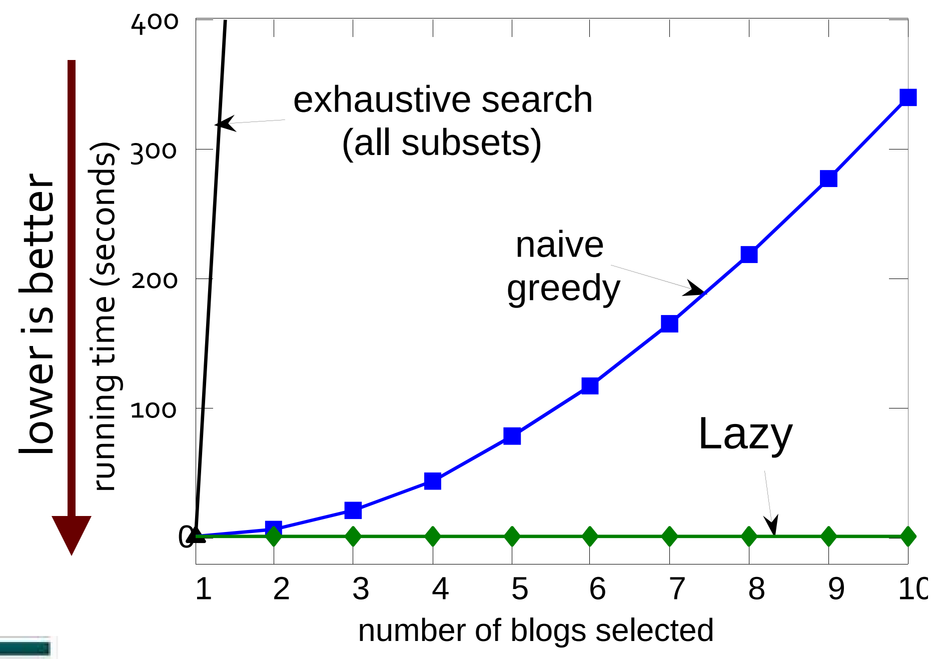
Detecting information outbreaks



Performance on Blog selection



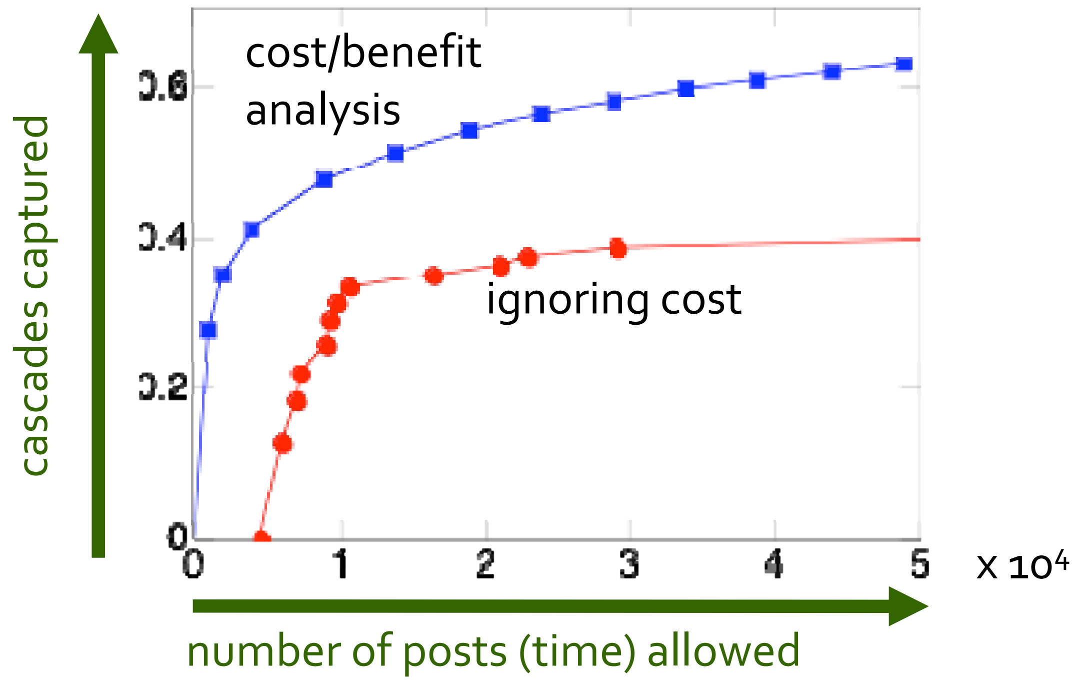
blog selection ~45k blogs



- Submodular formulation outperforms heuristics
- 700x speedup using lazy evaluations

Taking "attention" into account

- Naïve approach: Just pick 10 best blogs
- Selects big, well known blogs (Instapundit, etc.)
- These contain many posts, take long to read!



Minimization vs. Maximization

- Maximization of submodular functions:
  - NP hard
  - But can use greedy hill climbing to get ~63% of OPT
- Minimization of submodular functions:
  - Polynomial time solvable
  - Best known algorithm: $\Omega(n^5)$ function evaluations

Super- and Sub-modularity

- Set function F on V is called submodular if
  1) For all $A,B \subseteq V$:
  $F(A)+F(B) \geq F(A \cup B)+F(A \cap B)$
  $\Leftrightarrow$ 2) For all $A \subseteq B$, $s \notin B$:
  $F(A \cup \{s\}) - F(A) \geq F(B \cup \{s\}) - F(B)$

- F is called supermodular if -F is submodular
- F is called modular if F is both sub- and supermodular:
  E.g., for modular ("additive") F
  $F(A) = \sum_{i \in A} w(i)$

Other settings:

- Optimize the worst case:

  $$\mathcal{A}^* = \operatorname*{argmax}_{|\mathcal{A}| \leq k} \min_i F_i(\mathcal{A})$$

  - [Krause et al. '07]
- Online maximization of submodular functions:
  - [Golovin-Streeter '08]

Pick sets	A_1	A_2	A_3	…	A_T
SFs	F_1	F_2	F_3	…	F_T
Reward	$r_1=F_1(A_1)$	r_2	r_3	…	r_T

Total: $\sum_t r_t \rightarrow \max$

$\longrightarrow$ Time

Acks

- Most of the slides borrowed from Andreas Krause
- http://www.blogcascades.org
- http://www.submodularity.org